

Business Cycles, Bifurcations and Chaos in a Neo-Classical Model with Investment Dynamics

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Abstract

This paper is motivated by the rising interest in assessing the effect of disruptions in resources and environmental conditions on economic growth. Such an assessment requires, ultimately, the use of truly integrated models of the climate and economic systems. For these purposes, we have developed a Non-Equilibrium Dynamic Model (NEDyM) by introducing investment dynamics and nonequilibrium effects into a Solow growth model. NEDyM can reproduce various economic regimes, such as manager- or shareholder-driven economies, and permits one to examine the effects of disruptions on the economy, given either an assumption of steady-state growth or an assumption of business cycles with transient disequilibrium. We have applied NEDyM to an idealized economy that resembles in certain respects the 15-state European Union in 2001.

The key parameter in NEDyM is investment flexibility. For certain values of this parameter, the model reproduces classical business cycles with realistic characteristics; in particular, NEDyM captures the cycles' asymmetry, with a longer growth phase and more rapid contraction. The cyclical behavior is due to the investment-profit instability and is constrained by the increase in labor costs and the inertia of production capacity. For somewhat greater investment flexibility, the model exhibits chaotic behavior, because a new constraint intervenes, namely limited investment capacity. The preliminary results presented here show that complex behavior in the economic system may be due entirely, or at least largely, to deterministic, intrinsic factors, even if the economic long-term equilibrium is neo-classical in nature. In the chaotic regime, moreover, slight shocks – such as those due to natural or man-made catastrophes – may lead to significant changes in the economic system.

1 Introduction and motivation

This paper introduces a modeling framework for macroeconomic growth dynamics that is motivated by recent attempts to formulate and study “integrated models” of the coupling between natural and socio-economic phenomena. These attempts are driven, at least in part, by public debate about global issues, such as anthropogenic climate change. The challenge is to describe the interfaces between human activities and the functioning of the earth system over the very long term. In this context, economists have used primarily long-term growth models in the Solow tradition, relying on the idea that, over time scales of decades to centuries, the golden-age paradigm is an acceptable metaphor.

This approach appears, however, to be increasingly at variance with the nature of the policy debates in the field. Advocates of stringent emission limits are concerned about the cost of damages caused by climate change, while their opponents worry about the cost of greenhouse gas abatement. But balanced growth models that incorporate many sources of flexibility tend to suggest that the damages caused by disruptions of the natural — *i.e.*, physical and biological — planetary systems, as well as the mitigation policies proposed to prevent these disruptions, will entail only “a few percent” of losses in gross domestic product (GDP) over this century (IPCC, 2001). Both categories of activists tend thus to suspect that the figures suggested by current models underestimate either category of costs, since real economies rarely manifest a tendency to steady-state behavior.

Our initial motivation was thus to study the economic implications of a given climatic-change impact or of a proposed climate-related policy, in the context of two types of views of economic growth: one that postulates a steady-state economy, with no fundamental transition problems; the other that assumes intervals of fragility due to the turbulent conditions of a more-or-less regular economic cycle. This approach raises a set of scientific problems that are familiar to economic modelers, but in a novel context.

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The core debate in macroeconomics used to be whether governments could control business cycles in an efficient manner by manipulating fiscal and monetary policy. Instead, the focus here is on whether economic shocks – such as those caused by climate damages or by the modification of investment patterns for long-lived infrastructures to achieve mandatory decarbonization policies – will, or could, influence long-term growth pathways and generate a permanent and sizable loss of welfare. We venture to suggest that this novel approach leads to a modeling framework of considerable interest beyond its originally intended field of application. The initial results presented in this paper indicate that our framework helps to capture the role of institutional and technological inertia as key parameters that may control endogenous business cycles and even give rise to bifurcations in long-term growth trends.

The next section places our contribution in the context of existing modeling approaches to business cycles. Section 3 describes our non-equilibrium dynamic model NEDyM and the range of behavioral assumptions it can describe. Section 4 presents the basic behavior of NEDyM solutions for one of these options, namely the managerial economy with imperfect foresight; the equilibrium solution for this option is calibrated to roughly match the aggregate indicators of the European Union’s economy for 2001, and the periodic solutions are compared to standard business-cycle descriptions. In Section 5 we investigate more generally how solution behavior changes as the model’s key parameter, which governs investment dynamics, changes. These results are interpreted in terms of a systematic bifurcation analysis that leads from equilibrium to cyclic and on to chaotic behavior. In the last section we draw tentative conclusions and provide suggestions for future research.

2 The role of inertia in endogenous business cycles

The fact that major economic indicators — like production, prices, wages and consumption — undergo more-or-less regular ups and downs remains a major challenge for economic theory. One of the main trains of thought (*e.g.*, Slutsky, 1927; Frisch, 1933; Lucas, 1975; Kydland and Prescott, 1982; King and Watson, 1996; Wang and Wen, 2004), is that the economic system has a single stable equilibrium and cycles result from random shocks on money supply, technology or productivity; these shocks are progressively absorbed by the economic system, which returns to its long-term equilibrium.

Another strand of literature, stemming from Keynes and Kalecki, uses non-linear deterministic relationships between economic aggregates: starting from the Harrod (1939) model, which shows that growth becomes unstable if an investment accelerator is introduced, Samuelson (1939) demonstrated that this instability can be responsible for business cycles, while Kalecki (1937) obtained

cyclical dynamics by introducing a delayed dependence of investments on past profits into a multiplier framework. Solow (1956) contested the “*knife-edged*” equilibrium path of Harrod, where a small sidestep could lead to disaster. His argument relied on allowing substitution between capital and labor to restore the possibility of equilibrium growth pathways.

Arrow (1989) and others emphasized, however, the lack of evidence for the existence of such a stable equilibrium. In his Nobel Prize lecture, Solow (1988) stressed that, after a perturbation “*the economy that once strays from equilibrium growth [may] not automatically find its way back to any equilibrium path*”. His diagnosis was that “*Growth theory was invented to provide a systematic way to talk about and compare equilibrium paths for the economy. In that task it succeeded relatively well. In doing so, however, it failed to come to grips adequately with the right way to deal with deviations from equilibrium growth*”.

Confrontations and hybridizations between balanced-growth theories and business-cycle theories lead to many ways of obtaining endogenous economic cycles: financial constraints and changes in income (Kaldor, 1940; Hicks, 1950); discontinuous shifts in investment in response to differences between a desired and real level of capital stock (Goodwin, 1951); changes in income distribution and the role of the destruction of the *reserve army of labor* in a Volterra-type predator-and-prey model (Goodwin, 1967); and various Keynesian feedback channels (Chiarella and Flaschel, 2000; Chiarella et al., 2002; Asada et al., 2004). In neoclassical optimal growth models, perfect market clearing and perfect foresight rule out demand and income distribution effects. Still, even in such models, endogenous cycles can arise from savings behavior (Day, 1982) and from interactions between overlapping generations (Gale, 1973; Reichlin, 1986); distinction between economic sectors (Benhabib and Nishimura, 1979) or between wealth effects and real-interest-rate movements (Grandmont, 1985). In some of these models, either Keynesian or neoclassical, business cycles can even lead to chaotic behavior (*e.g.*, Day and Shafer, 1985; Chiarella, 1988); Rosser (1999) provides a review of the applications of chaos and complexity theory to economics.

Our contribution to these diverse attempts to model endogenous cycles and growth patterns arises from our focusing on the inertia of economic systems. Accounting for inertia is critical for the study of sustainability issues, because it acts as a cost-multiplier of the adaptation to new information (Ha-Duong et al., 1997). The concept of inertia allows one, furthermore, to capture in a very compact way the frictions of the social machinery in which Solow saw one major reason for disequilibrium growth¹. Into this category fall the con-

¹ “The markets for goods and labor look to me like imperfect pieces of social machinery with important institutional peculiarities. They do not seem to behave at all like transparent and frictionless mechanisms for converting the consumption and

straints on equipment turnover, especially in the infrastructure sector; the progressive deployment of innovations; and all the social constraints that delay the immediate implementation of investment or consumer decisions.

An additional interest of our approach is that it allows, as we shall see, the development of a modeling framework capable of representing various views of long-term economic growth. In particular, balanced-growth pathways can be reproduced, because we preserve Solow’s view that capital and labor can be substituted for each other in the macroeconomic production function, and thus avoid Harrod’s knife-edged behavior. Transient behavior and endogenous business cycle can arise, on the other hand, since we add to Solow’s framework the following two inertia-related effects:

- (1) Delays in the mutual adjustments between production and demand or capital and labor; these delays may be due to regulation, but also to the inertia of institutional and technical systems including the deployment of research and development results.
- (2) A Kalecki-type model of investment decisions which allows one to represent possible suboptimalities in the decision process, due to possible gaps between short-term signals and long-term economic circumstances.

3 A non-equilibrium dynamic model (NEDyM)

NEDyM is a highly idealized macro-economic model that follows the classical Solow growth model in considering an economy with one representative producer, one representative consumer and one good, used both for consumption and investment.

The original Solow (1956) model is composed of a static core describing the market equilibrium, and a dynamic relationship describing the productive capital evolution. In NEDyM, we translate the static core into dynamic laws of evolution by building delays into the pathways toward equilibrium. This device introduces short-term dynamics into the model.

3.1 *Variables and parameters*

The 8 state variables, whose evolution is described by ordinary differential equations (ODEs), are listed in Table 1. The 11 other variables, which are connected to the 8 state variables by algebraic equations, are listed in Table 2. In the climatological (Ghil and Childress, 1987) and meteorological (Kalnay, ¹leisure desires of households into production and employment” (Solow, 1988, p.311).

Symbol	Description	Unit
F	liquid assets of banks and companies	monetary
G	goods inventory	physical
K	capital	physical
L	number of employed workers	millions of workers
M	consumer money stock	monetary
p	price	monetary
w	wages	monetary
Γ_{inv}	producer investment ratio	no unit

Table 1

List of the model state variables (prognostic variables).

Symbol	Description	Unit
C	consumer consumption	physical
D	total demand (=sales)	physical
Div	dividends	monetary
\widetilde{Div}	expected future dividends	monetary
I	investment	physical
L_d	optimal labor demand (i.e. without demand constraint)	no unit
L_e	effective labor demand	no unit
S	available savings	monetary
Y	production	physical
Y_d	optimal production corresponding to L_d	no unit
Π	gross profits	monetary
Π_n	net profits	monetary
$\widetilde{\Pi}_n$	expected future net profits	monetary

Table 2

List of the other model variables (diagnostic variables).

2003) literature, it is common to call the first type of variables prognostic and the second diagnostic. The model parameters are listed in Table 3 and their values are justified in Section 4.1, where model calibration is discussed.

Symbol	Description	Value
A	total productivity	$A = 3.2 \cdot 10^{-4}$
e_{full}	equilibrium rate of employment	$e_{full} = 90\%$
L_{max}	total number of workers (in millions)	$L_{max} = 180$
γ_{save}	consumer savings ratio	$\gamma_{save} = 0.3$
ν	financial standard of profitability	$\nu = 3\% \cdot yr^{-1}$
ρ	required return on equity	$\rho = 10\% \cdot yr^{-1}$
τ_{dep}	capital depreciation characteristic time	$\tau_{dep} = 20$ years
α_{inv}	producer investment coefficient	varying
α_F	using rate of the producer liquid assets	$\alpha_F = 0.2 yr^{-1}$
α_M	using rate of the consumer money stock	$\alpha_M = 0.2 yr^{-1}$
α_p	price coefficient	$\alpha_p = 3.6 \cdot 10^{-3}$
τ_{empl}	employment characteristic time	$\tau_{empl} = 2$ years
τ_{wage}	wage characteristic time	$\tau_{wage} = 2$ years

Table 3

List of the model parameter values. The first 7 parameters determine the equilibrium of the model and are calibrated; the last 6 do not influence this equilibrium, only the model dynamics. The standard values of the latter, as used in Section 4, are shown in this table and are chosen in an *ad hoc* manner; α_{inv} will be varied systematically in Section 5.

3.2 Equations

This section describes how equilibrium constraints between variables are transformed into dynamic relations, thus increasing the number of prognostic variables of our model and hence its variability.

3.2.1 Goods market

In the Solow model, the price p is determined by the equality of production Y and demand D , $Y = D$. In NEDyM, $Y \neq D$ and a goods inventory G is introduced, filled by Y and emptied by D . At any time t , Y can differ from D : temporary overproduction or underproduction is possible. The dynamics of the goods inventory is driven by the difference between production and demand. It can be either positive or negative. It should be interpreted as a *selling lag* (or the opposite of a *delivery lag*): a positive value refers to temporary overproduction and can be interpreted as the time necessary to a producer to sell the goods he produced. A negative value refers to underproduction and can

be interpreted as the time it takes a consumer to get the goods he ordered². Underproduction can come from the technical lag to produce, transport and distribute goods and from possible undercapacity due to the inertia to build up new capacities.

Price increases or decreases as a function of the goods inventory situation, tending to return the state of the model to the equilibrium $Y = D$ with a null goods inventory ($G = 0$). This relaxation toward equilibrium guarantees the long-term “conservation of mass” in the model: any good that is produced is eventually sold, any good that is bought is eventually produced. As a consequence, the equality of production and demand is verified over the long term, but the delays in price adjustment can break this equality in the short term.

- Demand D equals the sum of consumption C and investment I :

$$D = C + I . \quad (1)$$

This is what we call a diagnostic, or algebraic, relation in our model.

- Goods inventory changes are driven by the difference between production and demand:

$$\frac{dG}{dt} = Y - D . \quad (2)$$

We call this a prognostic, or differential, relation

- Price evolution:

$$\frac{dp}{dt} = -p \cdot \alpha_p \cdot \frac{G}{D} . \quad (3)$$

Price changes are driven by the goods inventory state, with a given characteristic time allowing to account for an inertia in the price adjustment³. If the inventory is positive, it means that the production is not sold (or is sold after a delay), and that the market power is of the side of the buyers who can make the price decrease. If the inventory is negative, goods have to be produced and the price will increase. The inventory is compared with the demand to determine the price variation. Note that a possible asymmetry in the price behavior could be implemented at a latter stage.

² This also allows to account for services, which are a large part of the current economy and which cannot be stocked, like goods.

³ This price evolution rule differs slightly from the model of Calvo (1983) and of King and Watson (1996), who introduce inertia through a static, monopolistic competition model and a “*target*” price calculated by a fixed markup over the marginal cost. Here we aim at representing the adjustment process of the price in response to changes in production or demand.

3.2.2 Production function

As in the Solow (1956) model, we use a Cobb-Douglas function (Cobb and Douglas, 1928):

$$Y = f(L, K) = A \cdot L^\lambda \cdot K^\mu , \quad (4)$$

where $\lambda = 2/3$, $\mu = 1/3$, and A is the total productivity. The algebraic relation (4) gives the production as a function of the productive capital K and the number of employed workers L . Note that the productive capital is considered as a physical object (machines, plants, infrastructures...) not only as a sum of monetary investments.

3.2.3 Labor market

In the Solow model, w is such that $L = e_{full} \cdot L_{max}$: the economy is always at full employment. NEDyM models instead the producer as setting an effective labor demand L_e that would maximize its profits, as a function of the price and the wages; the latter are flexible over the long term and rigid over the short term. The number of employed workers L is driven by this effective labor demand with a delay. If labor demand is higher (respectively lower) than the equilibrium level $e_{full} \cdot L_{max}$, the wage increases (respectively decreases), so as to drive the employment level back to its equilibrium value.

- Employment rate evolution:

$$\frac{dL}{dt} = \frac{-1}{\tau_{empl}} (L - L_e) . \quad (5)$$

The producer adjusts the number L of workers in order to attain his effective labor demand L_e . Technical constraints, however, prevent an instantaneous adjustment process: changing the labor/capital ratio requires the producer to adapt the working organization and, possibly, even the productive capital has to be changed. Additionally, worker protection laws, administrative tasks, transaction costs and the search for qualified workers are also involved in the adjustment process. This is taken into account through the parameter τ_{empl} , which is the e-folding time of the convergence of L toward L_e .

- The evolution of nominal wages follows a Phillips (1958) curve:

$$\frac{dw}{dt} = \frac{w}{\tau_{wage}} \left(\frac{L}{L_{max}} - e_{full} \right) . \quad (6)$$

Following Rose (1967), wages are solely determined by the employment rate, with a given characteristic time. Calibrating τ_{wage} allows NEDyM to account for rigidities in the labor market. Wages increase when the employment rate is high, because of the corresponding negotiating power of the workers, individually or through trade unions.

At this stage no prevailing inflation climate nor price inflation expectations are taken into account, in spite of their potential importance (*e.g.*, Chiarella and Flaschel, 2000). Additional factors that may affect the differential relation (6) need eventually to be discussed because the social organization, the institutional conditions, the worker protection laws and agreements between social groups play an important role in determining wages.

- Producer labor demand optimization: The producer aims at optimizing his labor demand L_d , assuming that his production will be sold and equating the labor marginal productivity to the real wage. This optimization is modeled by the diagnostic relation:

$$\frac{w}{p} = \frac{df}{dL}(L_d, K) . \quad (7)$$

- Effective employment demand:

$$L_e = \text{Min}(L_{max}, L_d) . \quad (8)$$

3.2.4 Consumer behavior

In the Solow model, total income from wages and profits, *i.e.* $(wL + \Pi)$, is always equal to consumption plus savings, *i.e.* $(pC + S)$. In NEDyM, the consumer has an income of $(wL + Div)$, wages plus dividends, and can consume C ; in addition he can save, either by stocking (in his stock of money M) or by making this savings available for investment (S) through the purchase of new equity (for more details, see below the *Producer behavior* section).

- Consumer stock of money:

$$\frac{dM}{dt} = (w \cdot L + Div) - (p \cdot C + S) . \quad (9)$$

This prognostic rule for stock evolution introduces a delay in consumer behavior: an increase in his income will increase progressively consumption and available savings.

- Consumer consumption :

$$C = (1 - \gamma_{save}) \cdot \frac{1}{p} \cdot \alpha_M \cdot M . \quad (10)$$

At each instant, the consumer “spends” (*i.e.* consumption plus available savings) a constant part of his stock of money, $\alpha_M M$. This relation introduces a delay in the behavior of the consumer with respect to the change in his income. Moreover, it means that, when his income increases, his equilibrium stock of money increases also, so as to take into account the need for an increased stock of money to buy more goods. As in the Solow (1956)

model, the distribution between available savings S and consumption C is fixed exogenously by γ_{save} ⁴.

- Consumer savings:

$$S = \gamma_{save} \cdot \alpha_M \cdot M . \quad (11)$$

Consumer consumption and savings are therewith both modeled in NEDyM by diagnostic relations. Physical investment and consumption are measured in physical units, since they represent bought goods, while available savings is measured in monetary units. The redistributed dividends Div will be modeled in connection with producer behavior.

3.2.5 Productive capital

The productive capital K evolves here prognostically, as in the classical Solow (1956) model:

$$\frac{dK}{dt} = \frac{-1}{\tau_{dep}} K + I . \quad (12)$$

3.2.6 Producer behavior

In the Solow model, sales pD equal wages wL plus profits Π , and investment I equals savings S . A key feature of NEDyM is its introducing of an investment module, inspired by the Kalecki (1937) models, which forces to make explicit the assumptions about investment behaviors and to account for possible alternative regimes, a question which does not arise in the Solowian models. We introduce a stock of liquid assets F of companies, which is modeled prognostically; here F includes both the producer's stock of money and that of financial intermediaries like banks.

The stock F is filled by gross profits Π and by the available savings S from consumers. We assume that the consumers' savings S are made available to companies by sales of new equity, directly or through financial intermediaries. Assuming that the Modigliani-Miller theorem's hypotheses are verified (Modigliani and Miller, 1958), adding debts and loans does not change the investment problem.

At each time, a fixed part $\alpha_F F$ of F is used for physical investment and for distributing dividends. An investment ratio Γ_{inv} is used to distribute $\alpha_F F$ between physical investment I and dividends Div .

⁴ The consumer savings ratio γ_{save} is applied here to the consumer stock of money, not to the consumer income, and is thus not directly comparable with the classically defined savings ratio (*e.g.*, Solow, 1956).

Dividends Div are that part of the company's liquid assets that is redistributed to the consumers, but one key feature of NEDyM is that these dividends represent more than redistributed dividends *per se*: they include redistributed dividends, but also sales of assets, capital gains, spinoffs to shareholders, repurchase of shares, payments in liquidation, payoffs resulting from merger or acquisition, and awards in shareholders' lawsuits. The equivalence between all these categories is comprehensively demonstrated in Copeland and Weston (2003). This NEDyM modeling choice is justified by the fact that when there is higher financial investment than physical investment, the additional part goes back to the shareholders either as an increase in dividends or as a consequence of share spinoffs.

We translate here the fact that if the producer does not want to invest, it does not emit new shares and the savings that a consumer uses to buy shares (S) are only transferred to other consumers, *i.e.* to himself through Div in the model. This possibility is responsible for a non-productive closed loop of money, from the consumer stock of money, through savings to the producer stock of money, and back to the consumer stock of money through dividends. This closed loop can be interpreted as the process that changes, at equilibrium, the unrelated *ex ante* savings and investment into consistent *ex post* variables. Whatever the amount of available savings, if the economic situation is such that the capital profitability is low, these savings will not be transformed into physical investments but be redistributed through Div . Thus, the equality between savings and investment does not hold: savings and investment decisions are independent, at least to some extent. This loop has a very short characteristic time, of a few weeks, which explains the shortness of the model timestep (see below).

NEDyM assumes that physical investment I is driven, via Γ_{inv} , by the capital profitability, with an inertia that represents: (i) the fact that producers want to avoid uncertainties and adjustment costs (Day, 1979; Lucas, 1967) and thus do not react instantaneously to price signals; and (ii) the delay between the time an investment is decided upon and the time it is realized (Kalecki, 1937).

- Changes in the producer stock of liquid assets F are thus given by the differential relation:

$$\frac{dF}{dt} = \Pi + S - Div - pI . \quad (13)$$

- Gross profits Π are given by sales pD minus labor costs wL :

$$\Pi = pD - wL . \quad (14)$$

- Dividends and physical investment distribution follow the diagnostic relation:

$$pI + Div = \alpha_F F . \quad (15)$$

The producer uses a fixed amount of his stock of liquid assets for physical investment and dividends. The distribution between physical investment and dividends depends on the investment ratio Γ_{inv} :

$$I = \Gamma_{inv} \cdot \frac{1}{p} \cdot \alpha_F F . \quad (16)$$

- Producer investment ratio:

We model investment as a function of profitability (Kalecki, 1937; Rose, 1967; Malinvaud, 1982), and not as a function of the aggregated demand level, as in investment accelerator models (*e.g.*, Harrod, 1939; Goodwin, 1951; Schinasi, 1981). This modeling choice allows us to examine the consequences of a large range of decision-making behavior. This range can be described in a plane spanned by two axes.

The first axis marks the possible weights given to the two opposite views on agents' expectations, myopic and rational. Covering this wide range of options would have blurred our intended analysis of the endogenous mechanisms behind economic cycles, and so we retained, at this stage, the hypothesis that agents form their expectations based on observed values, either because they are myopic or because uncertainty makes current values be the best proxy for future values.

The second axis marks the possible weights given to the two opposite views on who are the decision makers in companies. At one end of the scale are the Schumpeterian entrepreneurs and Galbraithian managers, whose common goal is to maximize the long-term growth of the firm. In this case, the producer redistributes as dividends only the amount that is still available after the profitable investments have been funded. At the other end are the shareholders, who force the management of the firm to invest only what is left after the distribution of dividends, which have to yield a given level of return on equity. This behavior derives from one of the trends in economic organization since the mid-1980s, with company control edging more and more towards the shareholders, through institutional investors (*e.g.*, Jensen, 1986).

This second axis is critical for discussing institutional issues related to long-lived investments, and so we followed the distinction proposed by Aglietta and Rebérioux (2004) between a managerial economy and a shareholder economy. This distinction is obviously a caricature of opposite views of the economic system, both in actual terms (how does the system really function?) and in normative terms (what is an optimal balance between the power of the managers and of the shareholders?). But it has the advantage of being easy to translate into modeling terms that are clear enough to help determine how assumptions about decision-making behavior and institutional organization may impact economic cycles.

We now describe how each of these two options is represented in our modeling framework:

(1) Managerial economy:

The producer's net profit Π_n follows the *accounting definition of profit* (Copeland and Weston, 2003), *i.e.* gross profits minus capital depreciation:

$$\Pi_n = \Pi - \frac{1}{\tau_d} pK = pD - wL - \frac{1}{\tau_d} pK, \quad (17)$$

and the investment ratio follows the prognostic rule:

$$\frac{d\Gamma_{inv}}{dt} = \begin{cases} \alpha_{inv}(\gamma_{max} - \Gamma_{inv}) \left(\frac{\tilde{\Pi}_n}{pK} - \nu \right) & \text{if } \frac{\tilde{\Pi}_n}{pK} - \nu > 0 \\ \alpha_{inv}(\Gamma_{inv} - \gamma_{min}) \left(\frac{\tilde{\Pi}_n}{pK} - \nu \right) & \text{if } \frac{\tilde{\Pi}_n}{pK} - \nu \leq 0 \end{cases}. \quad (18)$$

The distribution between dividends and investment depends on the expected net profits per capital unit $\tilde{\Pi}_n$, compared with a *standard of profitability* ν . If the expected net profit per capital unit $\tilde{\Pi}_n/(pK)$ is higher than this standard, the producer increases his physical investments; if, on the contrary, the expected profit is lower than ν , investments are reduced. Assuming that observed values are the best guess of expected values at each point in time leads to:

$$\tilde{\Pi}_n = \Pi_n \quad (19)$$

The extrema $\gamma_{min} = 0$ and $\gamma_{max} = 0.8$ of Γ_{inv} are parameters that represent, respectively, the positivity of investment and the cash-flow constraint. Their values are arbitrary, to some extent, in the sense that somewhat different values would be compatible with an acceptable calibration of the model; but we checked, through sensitivity tests, that a reasonable range of γ_{min} and γ_{max} does not change qualitatively the model's behavior. Note that rule (18) behaves asymmetrically: for instance, the investment ratio Γ_{inv} can decrease more rapidly in response to a negative profitability signal when investment is high, because the factor $(\Gamma_{inv} - \gamma_{min})$ is large. On the contrary, the decrease is slowed down (respectively cancelled) when investment is already low (respectively null), because the factor $(\Gamma_{inv} - \gamma_{min})$ is close (respectively equal) to zero in this case.

The prognostic rule (18) can be interpreted using Tobin's (1969) Q : if the future net profit is assumed equal to the current one and if the company value V is equal to the discounted net profit flux, then $V = \Pi_n/\nu$, where ν here is the discount rate; Q is then defined as the company value divided by its capital replacement value pK . In this approach, Q drives the investment in the sense that investment progressively increases when $Q > 1$ and decreases when $Q < 1$.

The choice of ν determines the level of investment and thus the future production levels. In this article, ν is assumed to be constant at 3%. This value of ν is smaller than the currently observed financial standard of profitability, because the model does not separate between private-sector investments, decided solely on the basis of financial profitability,

and public-sector ones, which take additional considerations into account and have therefore, in general, lower internal return rates. Investment rule (18) is consistent with the functioning of the economy before the 1980s, as described by Berle and Means (1932).

(2) Shareholder economy:

In this case, the investment rule becomes

$$\frac{d\Gamma_{inv}}{dt} = \begin{cases} \alpha_{inv}(\gamma_{max} - \Gamma_{inv}) \cdot \left(\frac{\widetilde{Div+Err}}{pK} - \rho \right) & \text{if } \frac{\widetilde{Div+Err}}{pK} - \rho > 0 \\ \alpha_{inv}(\Gamma_{inv} - \gamma_{min}) \cdot \left(\frac{\widetilde{Div+Err}}{pK} - \rho \right) & \text{if } \frac{\widetilde{Div+Err}}{pK} - \rho \leq 0 \end{cases} \quad (20)$$

The allocation between dividends and investment depends here on the expected dividends per capital unit \widetilde{Div} . In this regime, if the expected dividends per capital unit are higher than the required return on equity ρ , the producer increases investments; in the opposite case, investments are reduced to the level that allows for redistributing the required amount of dividends. Again, expected dividends are supposed to be equal to observed dividends:

$$\widetilde{Div} = Div \quad (21)$$

To capture the difference between an economy subject to shareholder control and the previous managerial organization, it is, however, not sufficient to replace ν by ρ and $\widetilde{\Pi}_n$ by \widetilde{Div} . In fact, as already mentioned, the value of Div includes all investor gains, including capital gains and losses. This is why the variable Err has to be introduced to represent how unexpected variations in profits affect the equity value of companies. The variable Err translates the variations of equity values due to incomplete information in capital markets or formation of speculative bubbles. Assuming that Err is null amounts to assuming a “utopian shareholder economy,” in which markets operate so perfectly that shareholders fully incorporate all information in their evaluation of the expected value of an asset. This assumption is, obviously, at variance with the facts. Thus, for instance, during the 2001 crisis, a strong decrease in asset prices, which could be represented here by a strong decrease in Err , was in part compensated by higher dividends.

For the same reason that we did not explore here the whole range of decision behavior in a managerial economy, we will not elaborate in this paper on the consequences of various types of shareholder behavior. Preliminary results for a “utopian shareholder economy” show behavior that is less cyclic than in Sections 4.2 and 5 here, while the opposite result can be achieved in a shareholder economy with non-negligible values of Err .

A systematic comparison of how shareholder and managerial economies function, with respect to their business cycles and other characteristics, is a fundamental goal of our approach. This goal, however, cannot be achieved without a prior understanding of the mechanisms behind endoge-

nous cycles. We focus therefore in the rest of this paper on the analysis of the managerial economy, leaving the analysis of the shareholder version for a subsequent paper.

4 Model calibration and basic results

The model described in detail in Section 3 can be summarized in the equation

$$\frac{d\mathbf{X}}{dt} = N(\mathbf{X}) , \quad (22)$$

where the vector \mathbf{X} has as its components the 8 prognostic state variables listed in Table 1. Adding together Eqs. (9) and (13) shows that our NEDyM model obeys a conservation law for the amount of money, $M + F = \text{const.}$, since $D = C + I$. We verified, in fact, that the numerical simulations we carry out satisfy this conservation law, without having to impose it explicitly in the model⁵. This verification provides an independent check on the accuracy of our solutions.

4.1 Model equilibrium

When there is no investment dynamics, *i.e.* when $\alpha_{inv} = 0$ in Eq. (18) or (20), NEDyM has a stable equilibrium, $\mathbf{X} = \mathbf{X}_0$, which is still observed for sufficiently small values of α_{inv} (*e.g.* $\alpha_{inv} = 1.0$), whatever type of economy we consider, managerial or shareholder driven.

Six model parameters (A , e_{full} , L_{max} , γ_{save} , ν and τ_{dep} ; see Table 3) are chosen such that this equilibrium state is the 2001 economic state of the European Union (15 countries). This choice is made so as to give our model results a somewhat realistic flavor, even if there is no reason to believe that the European economy was at equilibrium in 2001. One should mention that the model is insensitive to changes in the monetary unit: if the price, the wages, the consumer stock of money and the stock of liquid asset of the producer are multiplied by a single number, the equilibrium and the dynamics are unchanged.

The dynamic parameters (α_{inv} , α_F , α_M , α_p , τ_{empl} , τ_{wage}) are not calibrated separately on historical data for several reasons: (i) because many variables are

⁵ In fact, the sum $M + F$ is constant to within a tolerance of 10^{-6} in the double-digit calculations of the numerical solutions, and there is no systematic drift.

not directly observable at all (e.g. G, K, L, or M); (ii) because the model has a very short time step, and very few data are available on such a time scale; and most importantly, (iii) because we do not pretend to reproduce historical data with such a simple, idealized model; we only try to reproduce the business cycle stylized facts (such as its asymmetric shape and characteristic lags between main economic variables) and to identify the driving mechanisms. Our NEDyM model is built to capture interactions between short-term and long-term economic behavior. Hence, reproducing the qualitative features of the cycle is more valuable at this point than reproducing historical data through an econometric calibration, which is only valid, in any case, over the short term.

Reasonable values are thus chosen in an *ad hoc* manner for five of the dynamic parameters, while the effect of α_{inv} on model behavior will be systematically studied in Section 5. In future work with more detailed versions of NEDyM, we expect to use the ideas of parameter estimation from the engineering and control literature (Jazwinski, 1970; Gelb, 1974; Kendrick, 2005) in order to determine the history of the system, as well as the dynamic parameter values, from the observable economic variables, when and where available. Such ideas have been successfully applied for very large systems with partial and irregular observations, in meteorology (Bengtsson et al., 1981; Kalnay, 2003) and oceanography (Ghil and Malanotte-Rizzoli, 1991; Malanotte-Rizzoli, 1996; Ghil, 1997).

All parameter values used in this section are listed in Table 3, except α_{inv} , which is set equal to 1.0. The corresponding equilibrium state is reproduced in Table 4, where it is compared with observed values for the European economy from Eurostat (2002).

We note that out of the available amount of money of 6 trillion euros (available savings S plus gross profits Π), 2 trillions are used for physical investment I and 4 trillions are redistributed as dividends Div (in the broad sense, *i.e.* including dividends, spinoff to shareholders, etc.). Moreover, total income (wages wL plus dividends Div) is 10 trillion euros, greater than GDP (9 trillion euros). This is explained by the fact that we do not consider *net flows* over one year (as the national accounting system does) but *oriented flows* (summed over one year). In NEDyM, as explained in section 3.2.6, there is a nonproductive closed loop from the consumer stock of money, through consumer savings, on to dividends and back to the consumer stock of money. This loop is not captured in the national accounting. If we calculated instead net flows that are comparable with the national accounting system, dividends (*ex ante* value) in the model are reduced to 3 trillion euros (*ex post* value) and consumer savings (*ex ante* value) are reduced to 2 trillion euros (*ex post* value), leading to a consistent equilibrium.

Symbol	Description	Equilibrium (and net flows)	2001 EU-15 observed values
Y	production (=demand=GDP)	9	8.8
L	number of employed workers	90%	92.6 %
wL	total annual wages	6	5.6
			(including taxes)
C	consumer consumption	7	6.8
S	consumer available savings	3 (2)	1.8
Π	gross profits	3	3.2
Div	annual dividends	4 (3)	3.2
I	physical investment	2	1.8

Table 4

NEDyM equilibrium state (when necessary the corresponding net flows comparable with the national accounting system are in parentheses) and EU-15 economic variables in 2001 according to Eurostat (2002). Every value is in trillions of euros.

With the parameter set given in Table 3, and $\alpha_{inv} = 1.0$, the equilibrium values of the NEDyM variables correspond to the steady state of a neo-classical Solow model with an equivalent savings ratio $\gamma_{save}^* = 22\%$.

The parameters that affect only the model dynamics (lower half of Table 3) can be interpreted as follows. With $\alpha_p = 3.6 \cdot 10^{-3}$, if the goods inventory is negative and fixed at minus 1 month of aggregated demand (a one-month delivery lag), the price diverges, following an exponential function of time, with a characteristic time of 10 years. With $\tau_{wage} = 2$ years, if there is a fixed 1% difference between the instantaneous employment rate and the equilibrium employment rate, the nominal wages diverge with a 2-year characteristic time. If a fixed 1% difference between $\Pi_n/(pK)$ and ν is imposed, the investment ratio Γ_{inv} tends to γ_{max} with a characteristic time (or e-folding time) equal to $100/\alpha_{inv}$ years. If $\alpha_{inv} = 1$, this characteristic time is about 100 years. If $\alpha_{inv} = 2.5$, this characteristic time is about 40 years and if $\alpha_{inv} = 10$, it is about 10 years. If $\alpha_{inv} > 100$, this characteristic time is lower than one year, Γ_{inv} shifts very quickly between γ_{min} and γ_{max} , and the situation is close to the Goodwin (1951) investment model, where investment can only have two values. Of course, if the difference between $\Pi_n/(pK)$ and ν is larger, the characteristic times are reduced accordingly.

In the previous subsection, we have seen that our NEDyM model exhibits, for reasonable parameter values, an equilibrium solution that resembles, in many respects, a fairly realistic, neo-classical equilibrium state. For the parameter values in Table 3, and with $\alpha_{inv} = 2.5$, this equilibrium solution still exists but is actually unstable. Changes in solution behavior and stability will be investigated in detail in Section 5.

The true interest of NEDyM, though, lies in its time-dependent behavior, which we start to explore in the present subsection. For the parameter values in Table 3, and with $\alpha_{inv} = 2.5$, our model possesses a stable periodic solution, with variables that oscillate around the equilibrium values listed in Table 4. This periodic solution is shown in Fig. 1.

The model oscillations in economic aggregates — profits, production and employment — exhibit several characteristics of the business cycles of the second part of the 20th century. The amplitude of the oscillation is unrealistic but its 5.4-year period is consistent with the mean business cycle period (see Zarnovitz, 1985; King and Watson, 1996; Kontolemis, 1997; Süßmuth, 2002, the National Bureau of Economic Research (NBER) website⁶). The observed variability in the period, however, is not reproduced in Fig. 1, since NEDyM produces only regular cycles for $\alpha_{inv} = 2.5$; see, however, the results for higher values of α_{inv} in Section 5.

The model cycle is composed of several phases, which are consistent with the cycle description in the review paper by Zarnovitz (1985), and whose durations are consistent with the data of Kontolemis (1997), Süßmuth (2002) and NBER:

- From $t = 0$ to $t = 1$ years, a *recovery phase*, during which the production increases with slight oscillations. This recovery phase is characterized by an increasing employment level, accompanied by increases in consumption and investments. This is responsible for an increase in total demand, which leads to a rise in profits (and price) and consequently feeds back on investments (and employment level). These positive feedback loops constitute the multiplier-accelerator effect (Harrod, 1939), which is a short-term Keynesian effect.
- From $t = 1$ to 3 years, an *expansion phase*, during which the employment rate is high and increases further, while the wages are rising; this lasting favorable development leads, however, to a growing inflation rate of price and wage that reduces the multiplier-accelerator effect on investment. A decrease in consumption is also observed because inflation reduces the pur-

⁶ <http://www.nber.org/cycles.html>

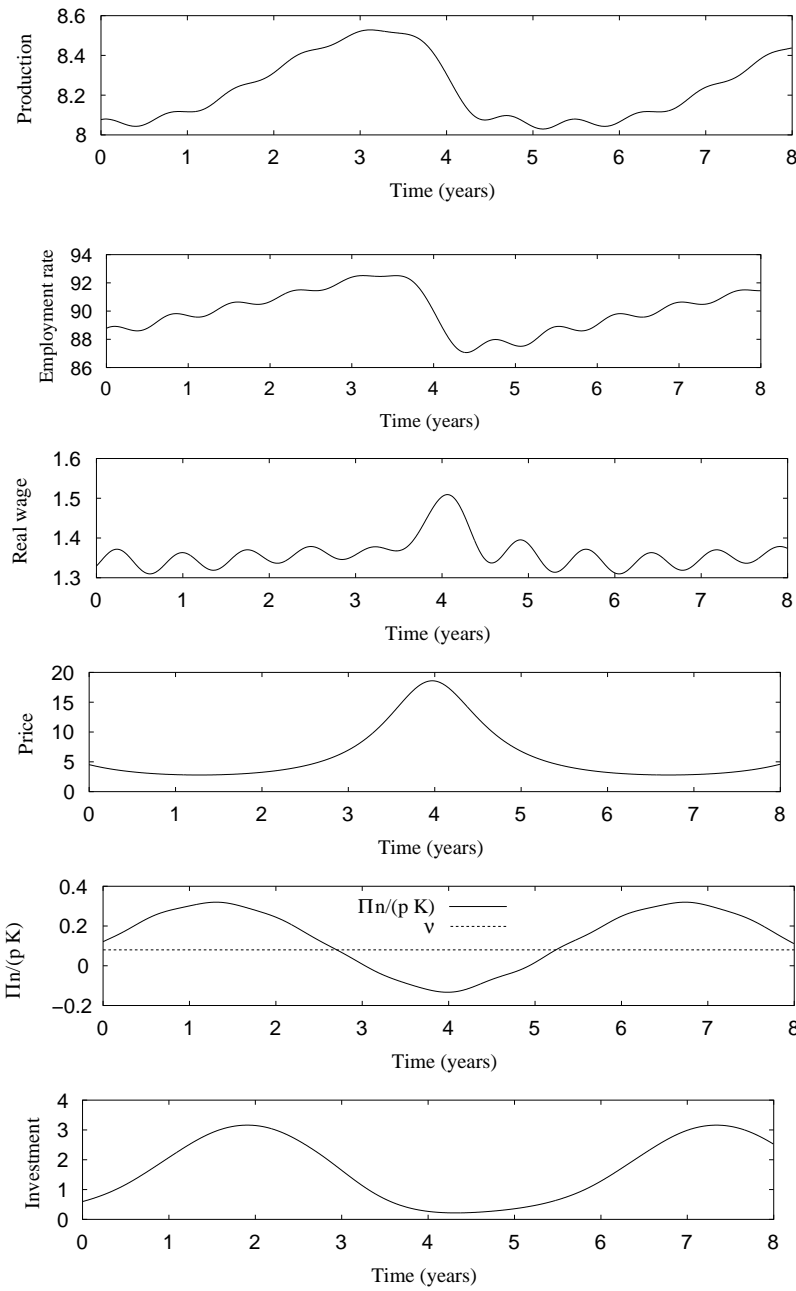


Fig. 1. Model business cycles for $\alpha_{inv} = 2.5$; for clarity, more than one full cycle is plotted. The variables represented are: (a) production Y ; (b) employment rate; (c) real wage w/p ; (d) price p ; (e) the net profits per capital unit $\Pi_n/(pK)$, with the financial standard of profitability ν (dotted); and (f) investment I .

chasing power of the consumer stock of money, such as in Pigou's (1947) effect, but the income from sales pD is approximately constant. During this phase, the net profits per capital unit, $\Pi_n/(pK)$, continuously decrease, because of the labor cost increases and because inflation increases the value of the capital (pK).

This phase of growing investment and increase in the circulation velocity

of money correspond: (i) to a period of Marxian “*destruction of the reserve army of labor*,” which is also responsible for the profit decrease in Goodwin (1967), Skott (1989) or Rose (1967); and (ii) to an inflation period, during which the price increase reduces the net profits per capital unit⁷. This reduction in profits is a classical effect, due to the labor availability constraint and to the production system inertia; it opposes the investment instability and is responsible for the cycle’s turning point.

- From $t = 3$ to 4.5 years, the *contraction phase*: at $t=3$ years, the net profit per capital unit $\Pi_n/(pK)$ falls below the financial standard ν and so the investment ratio begins to decrease, amplifying the demand decrease due to the inflation, and leading to a reduction in sales income (pD). Profits are thus brutally reduced. The economy reaches a situation of over-production, the goods inventory increases quickly, which causes the price to stabilize and then decrease from $t = 4$ years on. At that time, there is deflation: the price decreases and the employment rate decreases very quickly. The real wages increase during the boom and the early contraction, and begin to decrease during the late contraction and the depression. The profit per capital unit keeps decreasing during the early contraction and then begins to recover.
- From $t = 4.5$ years to $t = 5.2$ years, the *depression phase*: the investment ratio keeps decreasing but, because of the price decrease, the consumption begins to increase again. Moreover, for reasons that are the symmetric opposite of those in the expansion phase, the real wage, employment and price decrease restore the net profits per capital unit, until $\Pi_n/(pK)$ rises again above the financial standard of profitability at $t = 5.2$ years. At that point, the investment ratio rises again and the economy enters into a new recovery phase.

The NEDyM business cycle is thus consistent with the *stylized facts* explored by Zarnovitz (1985) (see in particular his Table 4). According to this author, the following relationships hold: the variables that are roughly coincident with the cycle are production Y , unemployment $(L_{full}-L)/L_{full}$; inflation $(dp/dt)/p$ and the circulation velocity of money. The variables that lead the cycle are the new orders for consumer goods and material, related here to consumption C ; the change in unfilled orders, related here to the time derivative of the goods inventory dG/dt ; the profit margins and the ratio of price to unit labor cost, related here to the gross profits Π . The variables that lag the cycle are the real wage w/p ; labor share in national income $wL/(wL + Div)$; and the inventories G .

These phase relationships are well reproduced by NEDyM, overall. Moreover, our model captures the co-movement of inflation and price with the cyclical

⁷ The price increase makes the profits rise, but it downsizes also the demand and increases the productive capital value. The total effect is a downsizing of the profit per capital unit.

component of production (e.g., King and Watson, 1996; Yun, 1996): inflation is procyclical and aggregate price countercyclical (but positively correlated with the lagged values of production), as observed in the data. This satisfying NEDyM behavior arises from nominal price rigidity⁸ and is consistent with the “*new Keynesian Phillips curve*” that establishes a positive link between inflation and some measure of overall activity.

Notice also that our model reproduces the observed asymmetry of business cycles (see NBER website or Kontolemis, 1997), with the recession phase in terms of production and employment much shorter than the expansion phase. This asymmetry is due to the following process: during the expansion phase, the economy shifts from overproduction to underproduction, and the goods inventory becomes negative, thus leading to price increases. The total demand is, however, high during this phase, and the amount of goods inventory stays moderate when compared with demand. As a consequence, the price responds only slowly. To the contrary, during the contraction phase, the total demand is very low. Thus, when the economy shifts back from underproduction to overproduction, the corresponding positive goods inventory becomes suddenly quite high, when compared with demand, and the price responds rapidly. This difference explains why the recession is more brutal than the recovery. It is noteworthy that the cycle asymmetry is positively correlated in our model with the cycle amplitude in terms of demand.

The main mismatches between our model cycle and the Zarnovitz (1985) stylized cycle are: (i) the fact that the total income $wL + Div$ is lagging in the model, unlike in reality, where it is coincident. This is due to the fact that the NEDyM oscillation in total labor costs wL has a much greater amplitude than the gross profits Π , which is unrealistic. (ii) The fact that the total sales decrease too early in the cycle and that the total sales oscillation is larger than the production oscillation. Both problems are linked to the amplitude of the price-and-wage oscillation being too large. A better calibration of our model, according to the parameter estimation ideas outlined in Section 4.1, could lead to an even better agreement with stylized business cycles, as described in the literature.

All along our model cycle, a shorter-period oscillation in wages and employment is observed, which also affects production; see especially panels (a)–(c) of Fig. 1. The period of this oscillation is of slightly less than a year. Macroeconomic dynamics is affected by several seasonal forcings (e.g., agricultural production, consumption patterns, etc; see Wen (2002)), which are not present in the current version of NEDyM. In a more realistic model ver-

⁸ Wang and Wen (2004) explain the same behavior by an endogenous monetary policy acting upon the illusion that prices are sticky in a flexible-price real-cycle model.

sion, in which seasonal effects would be present, this subannual oscillation might be frequency-locked to the seasonal cycle, and thus amplify the annual oscillation (Winfree, 1980; Ghil and Childress, 1987; Jin et al., 1994, 1996).

Notice that NEDyM is not Keynesian *per se*, since every behavioral relationship depends only on prices, even if the delays in price adjustment can be responsible for Keynesian behavior over the short term. In other words, prices and wages do not adjust instantaneously, but agent decisions (investment, labor demand, etc.) are still made as a function of these variables; there is no direct influence of quantities (*e.g.* goods inventory) on the decisions.

5 Time-dependent model behavior

It turns out that the behavior of such a simple dynamic model can be surprisingly rich. Rather than trying to calibrate the parameters in the second half of Table 3, we explore the model's parameter space. In particular, we focus on the dependence of model behavior upon the investment ratio flexibility α_{inv} .

5.1 Description of behavior types

If $\alpha_{inv} = 0$, the producer investment ratio Γ_{inv} stay constant, according to Eq. (18). The model possesses in this case the equilibrium described in Section 4.1 (see again Table 4), in which the employment rate is at its equilibrium value e_{full} , the goods inventory is null and the productive capital and the production depend on the investment ratio Γ_{inv} , which is fixed. This NEDyM steady state is equivalent to the steady state of a Solow model with a fixed savings ratio.

If the producer investment ratio is allowed to vary, several types of model behavior are possible:

- **Situation 1.** If the investment ratio is varying slowly, for example with an investment ratio coefficient $\alpha_{inv} = 1.0$, Γ_{inv} tends toward its equilibrium value, where the profit per capital unit is equal to the financial standard of profitability ν . The model thus reaches a stable equilibrium. In this case, ν drives directly the producer investment ratio and thus the level of productive capital and the economic activity. As described in Section 4.1, the NEDyM steady state is then that of a Solow (1956) model with a given savings ratio.
- **Situation 2.** If the investment ratio coefficient is higher, for example $\alpha_{inv} = 2.5$, the model does not reach a stable equilibrium any more but tends to

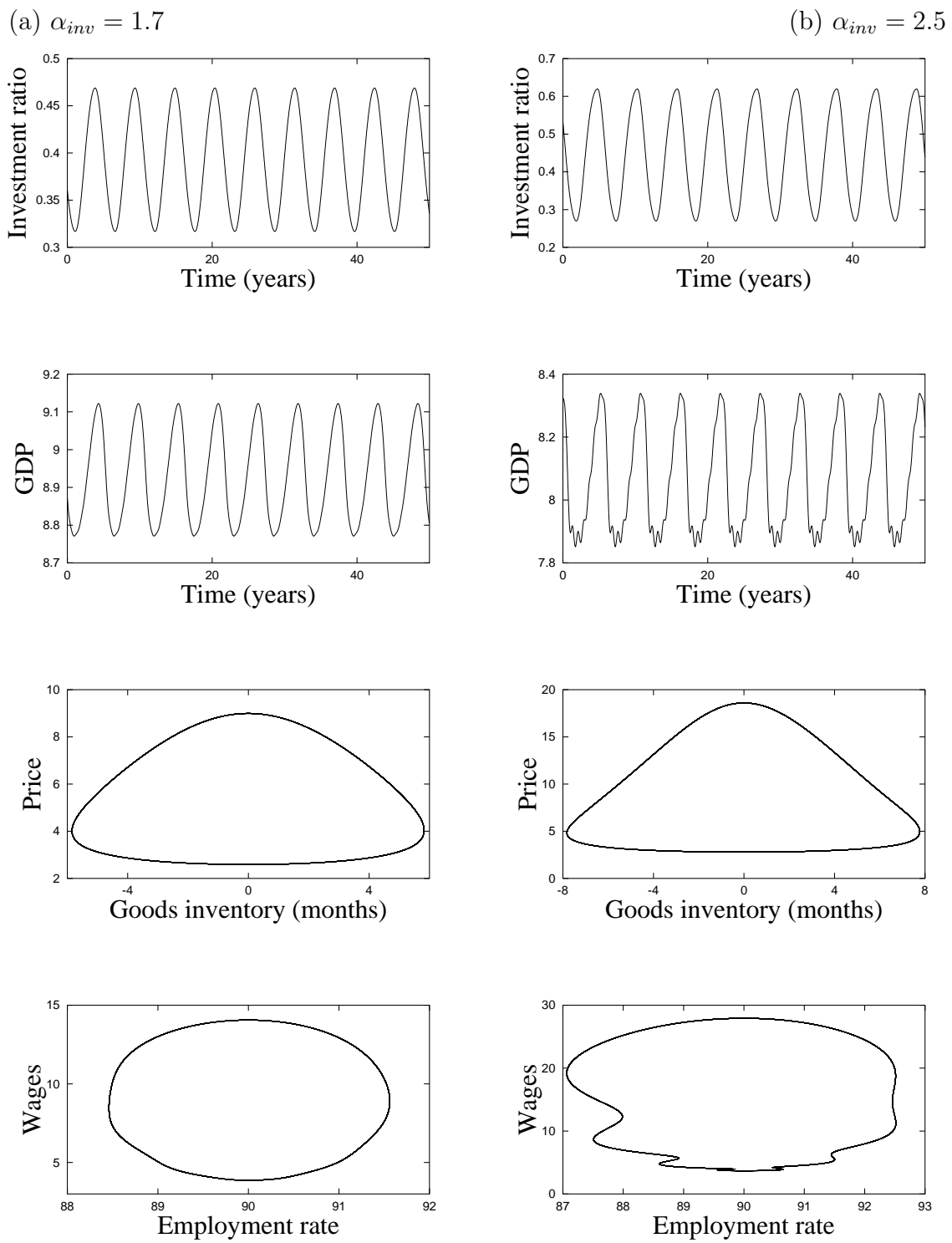


Fig. 2. Model behavior for (a) $\alpha_{inv} = 1.7$ (left column) and (b) $\alpha_{inv} = 2.5$ (right column). Notice the higher amplitude and greater irregularity of the oscillations in the right column.

an oscillatory solution, as described in Section 4.2 (see again Fig. 1). The oscillation in the investment ratio Γ_{inv} spreads into the production system, through the investment level. Figure 2 shows the evolution of the model state in this case, for two values of α_{inv} , after the initial transients die out.

The fundamental reason for this oscillatory behavior is that the investment increases when the demand is high and decreases when the demand is low; this response of the investment enhances the demand variations. Like in Harrod (1939), the multiplier-accelerator effect destabilizes the model. The reduction in demand, however, does not arise here from Keynesian money hoarding, but from changes in the amount of money involved in the nonproductive closed loop of money between savings and dividends: the saved money is not used for productive investments, because the low level of profits discourages producers from investing, but is redistributed through dividends.

The multiplier-accelerator instability is constrained in our model by two effects: (i) by the price increase that is due to the inertia of the supply side, which cannot respond to the increasing demand; and (ii) by the increase in total labor costs wL , due to the high employment and the wage increase. These processes reduce the profits per capital unit and oppose the investment instability. Because of the existence of these destabilizing and stabilizing processes, the model reaches an oscillatory behavior. The labor cost constraints are responsible for the turning point in many models (*e.g.*, Rose, 1967; Goodwin, 1967; Skott, 1989); in NEDyM these constraints do not act alone: the role of price inflation is also significant.

An additional feature of the oscillatory solutions in Figs. 1 and 2 is that the employment rate oscillates around its equilibrium value. This behavior is explained by the fact that wages are flexible over the long-term, in spite of inertia. The production, however, does not oscillate around the model's equilibrium value of 9 trillion euros, but around a suboptimal value, understood here as a value lower than the equilibrium value (see $\alpha_{inv} = 2.5$ in Fig. 2b). Comparing Fig. 2a with Fig. 2b, we see that the larger the oscillation's amplitude, the lower the mean state around which the production oscillates. This suboptimality is due to the nonlinearity of the model, especially of the Cobb-Douglas production function: a zero-mean oscillation in one variable can have significant consequences on the mean value of another variable. This “nonlinear rectification mechanism” makes it impossible to express the long-term behavior of our model through relationships between time-averaged values: modeling the short-term dynamics is necessary to understand the long-term behavior.

Not only do the model oscillations increase in amplitude as α_{inv} increases, but the solutions also become more irregular. In particular, the oscillation in employment rate and wages is quite smooth for $\alpha_{inv} = 1.7$ but develops kinks for $\alpha_{inv} = 2.5$. These kinks are due to the interaction that takes place between the wages (labor market) and the price (goods market) when the oscillation's amplitude is large enough; they affect the production pathway,

which exhibits marked irregularity, especially when production is lowest.

- **Situation 3.** For even higher values of α_{inv} , for example $\alpha_{inv} = 10$, the investment ratio Γ_{inv} may reach its upper or lower bounds, γ_{max} or γ_{min} , and new nonlinearities appear, cf. Eq. (18). These constraints can be understood as cash-flow constraints on investment and are imposed here in a rather crude manner. Such constraints arise, however, quite generally in dynamical systems, whether natural (physical or biological) or socio-economic, as a result of competition between intrinsic instabilities and limited resources. The limitations are typically on energy in physical systems and may be on labor, available production or natural resources in the socio-economic context.

Figure 3 shows the effect of these limitations on the long-term behavior of NEDyM solution for two values of α_{inv} . The new nonlinearities associated with the bounds on the investment ratio interact with those already present in Fig. 2. Together, they change qualitatively the model dynamics, which becomes chaotic.

In this case, the delayed coupling between a positive (profit–investment) and a negative (labor-cost–investment and price–investment) response is no longer alone in driving the oscillations. Because Γ_{inv} approaches its upper bound γ_{max} in Fig. 3a and both bounds in Fig. 3b, the oscillatory mechanism is further affected by two constraints: investment cannot be negative (no early capital retirement is allowed) and investment cannot exceed 80% of the available flows ($pI + Div = \alpha_F F$). The nonlinear effect of these constraints perturbs the regular oscillation and contributes to the chaotic behavior.

This behavior recalls the results of Day and Shafer (1985), who found chaotic trajectories in a Keynesian model when the *induced investment* is strong enough. Their induced investment strength can be related to the α_{inv} value in NEDyM.

5.2 Bifurcation analysis of model dynamics

We have seen, so far, in a general and qualitative way, that NEDyM solution behavior depends on the parameter values, and particularly on the value of the investment coefficient α_{inv} . Following the general approach of dynamical systems theory (Guckenheimer and Holmes, 1997), we study in the present section this dependence by tracking the changes in qualitative behavior of model solutions as α_{inv} increases. An application of this approach, called bifurcation analysis, to macroeconomic models appears, for instance, in Benhabib and Nishimura (1979), Barnett and He (2002) or Chiarella et al. (2002). Bifurcation theory has also been applied systematically to various areas of interest to macroeconomics, such as population dynamics (May, 1974) and climate dynamics (Ghil and Childress, 1987).

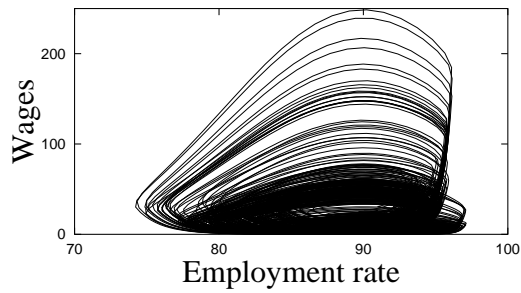
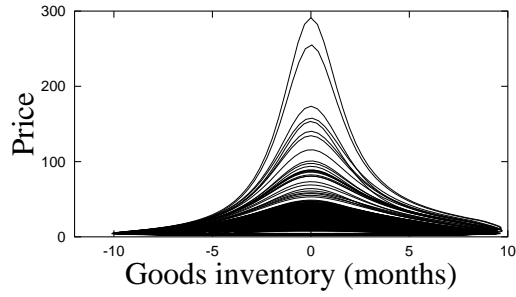
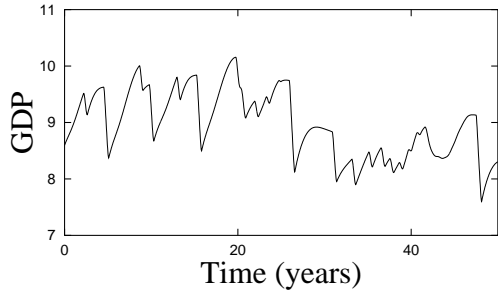
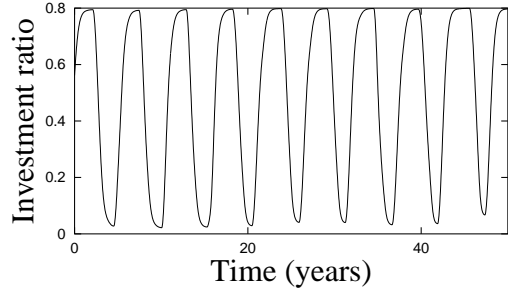
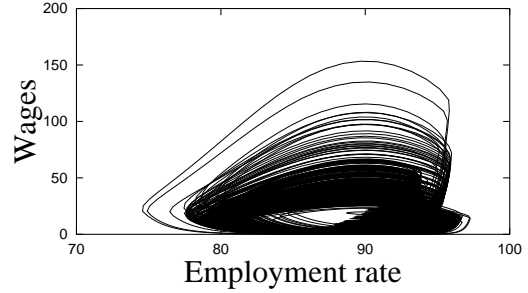
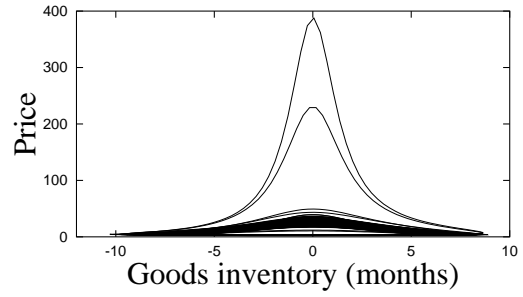
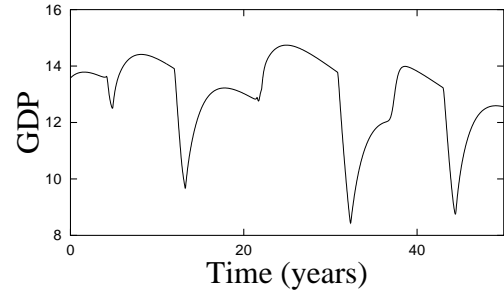
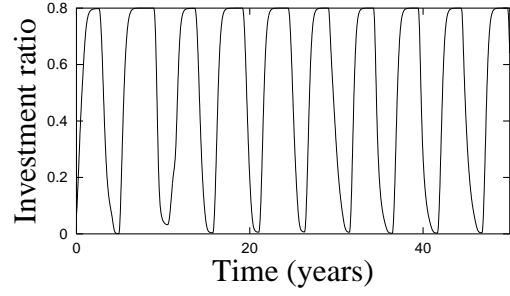
(a) $\alpha_{inv} = 10$ (b) $\alpha_{inv} = 20$ 

Fig. 3. Model behavior for (a) $\alpha_{inv} = 10$ (left column) and (b) $\alpha_{inv} = 20$ (right column). Notice that the investment ratio Γ_{inv} reaches its upper bound $\gamma_{max} = 0.8$ in the left column, while it reaches both γ_{max} and $\gamma_{min} = 0.0$ in the right column.

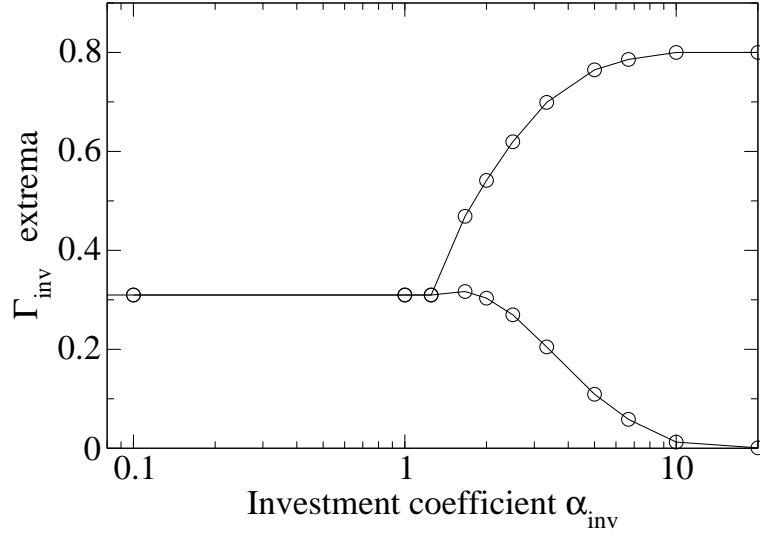


Fig. 4. Range of investment ratio values as a function of the investment coefficient α_{inv} .

Figure 4 shows the extrema of the investment ratio oscillation with respect to the investment coefficient α_{inv} . A bifurcation from stationary to time-dependent solutions is clearly visible for $1 < \alpha_{inv} < 2$. A careful analysis shows that the bifurcation occurs at $\alpha_{inv} \approx 1.4$: for $\alpha_{inv} < 1.4$, the investment ratio Γ_{inv} does not oscillate and tends to an equilibrium value; for $\alpha_{inv} > 1.4$, Γ_{inv} tends to a stable limit cycle, as shown in Sections 4.2 and 5.1 (see Figs. 2 and 3).

The transition from a stable equilibrium to a stable limit cycle is usually associated with a Hopf bifurcation (Guckenheimer and Holmes, 1997). To check whether this is the case here too, we linearized the model around the equilibrium state \mathbf{X}_0 , described in Section 4.1. The evolution of a perturbation $\delta\mathbf{X}$ around \mathbf{X}_0 is given by:

$$\frac{d\delta\mathbf{X}}{dt} = \mathbf{M}(\mathbf{X}_0, \alpha_{inv}) \delta\mathbf{X}, \quad (23)$$

where $\mathbf{M}(\mathbf{X}_0, \alpha_{inv})$ is the Jacobian matrix of the model at \mathbf{X}_0 , for a given value of α_{inv} ,

$$\mathbf{M}(\mathbf{X}_0, \alpha_{inv}) = \left. \frac{\partial \mathbf{N}}{\partial \mathbf{X}} \right|_{\mathbf{X}=\mathbf{X}_0}. \quad (24)$$

Since the model has 8 state variables (see Table 1), \mathbf{M} is an 8x8 matrix. The

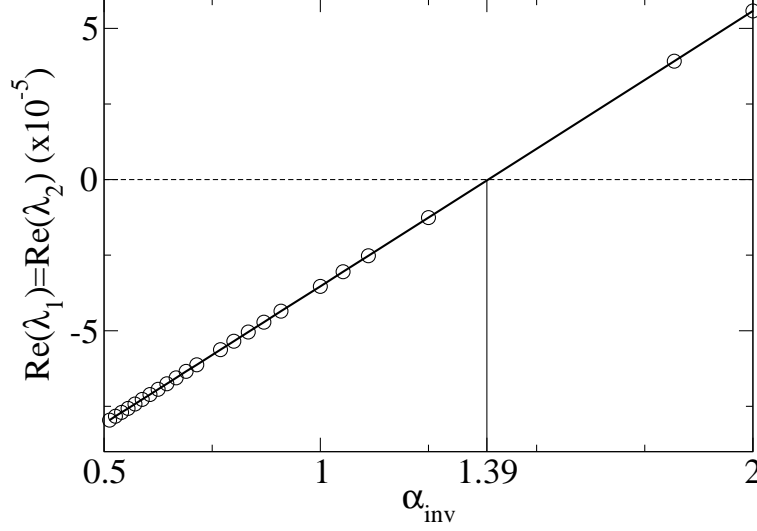


Fig. 5. Real part of the first pair of eigenvalues as a function of the investment coefficient α_{inv} . Open circles are the calculated values, while the solid line is a straight line through these values.

calculation of the eigenvalues λ_i ($i = 1, \dots, 8$) of $M(\mathbf{X}_0, \alpha_{inv})$, for various values of α_{inv} , gives the stability of the equilibrium and the model solutions' behavior around this equilibrium. The calculation was carried out for 25 values of α_{inv} , from 0.5 to 2.0. For all the values of α_{inv} used, we found four main eigenvalues: the other four are all 3 orders of magnitude smaller than the four main ones and have a negative real part.

The four main eigenvalues are distributed into two pairs of complex conjugate eigenvalues. The first pair (λ_1, λ_2) has an imaginary part that is independent of α_{inv} and equal to $-0.32 \cdot 10^{-2}$, which corresponds to an oscillation with a 5.5-year period, as observed in the model simulations (Figs. 1, 2, or 3). The real part depends on α_{inv} , as plotted in Fig. 5. This figure shows that the real part of λ_3 and λ_4 is a linear function of α_{inv} , at least in the range we studied. It follows that, for $\alpha_{inv} = 1.39$, the real parts of these eigenvalues are null and the equilibrium is neutral. When $\alpha_{inv} < 1.39$, the real parts are negative, the equilibrium is stable, and this pair of eigenvalues corresponds to a damped oscillation with a 5.5-year period. When $\alpha_{inv} > 1.39$, the real parts are positive, the equilibrium is unstable and this pair of eigenvalues corresponds to a growing oscillation. This oscillation is bounded and tends to a limit cycle, because of the model's nonlinearity. This behavior corresponds to a Hopf bifurcation of the model at $\alpha_{inv} = 1.39$.

The second pair (λ_3, λ_4) is independent of α_{inv} altogether: its real part is $-0.66 \cdot 10^{-3}$, which corresponds to an e -folding time of 4.15 yr; its imaginary

part is $-0.23 \cdot 10^{-1}$, corresponding to an oscillation with a 276-day period. This oscillation arises from labor-market dynamics and the Phillips curve, Eq. (6); it corresponds to the Goodwin (1967) cycle.

5.3 Lyapunov exponents

To verify that our model's behavior is indeed chaotic when the investment flexibility is high, we computed its Lyapunov exponents (Ghil and Childress, 1987; Guckenheimer and Holmes, 1997; Sprott, 2003). To do so, we introduce the state transition matrix Φ defined by:

$$\Delta \mathbf{X}(t) = \Phi(t, \tau) \Delta \mathbf{X}(\tau) , \quad (25)$$

where $\Delta \mathbf{X}(t)$ is a perturbation around a control trajectory $\mathbf{X}(t)$. In other terms, $\Phi(t, \tau)$ describes how a perturbation applied to the model state at time τ propagates to time t .

To calculate Φ , we integrate forward in time, in parallel with the state of the model, the equation:

$$\frac{d\Phi(t, \tau)}{dt} = \mathbf{M}(\mathbf{X}(t), \alpha_{inv}) \Phi(t, \tau) . \quad (26)$$

where $\mathbf{M}(\mathbf{X}(t), \alpha_{inv})$ is the Jacobian matrix of the model at $\mathbf{X}(t)$; see Eq. (24). A singular vector decomposition is then applied to $\Phi(t, \tau)$:

$$\Phi(t, \tau) = \mathbf{U} \mathbf{w} \mathbf{V}^+ ; \quad (27)$$

here \mathbf{w} is a diagonal matrix, with nonnegative diagonal values w_i , which depend on t and τ , while \mathbf{U} and \mathbf{V} are unitary matrices, which depend also on t and τ , and $()^+$ denotes the transpose. This equation can be rewritten as:

$$\Phi(t, \tau) = \mathbf{U} \mathbf{V}^+ \mathbf{V} \mathbf{w} \mathbf{V}^+ , \quad (28)$$

where $\mathbf{U} \mathbf{V}^+$ is a rotation matrix and $\mathbf{V} \mathbf{w} \mathbf{V}^+$ is a nonnegative semi-definite matrix.

Hence, the values $w_i(t, \tau)$ describe how a volume of perturbations applied at time τ to the model state is propagated by the system up to the time t , each w_i corresponding to one direction in phase space.

If all the $w_i(t, \tau)$ tend to zero, this volume decays with time to one point, any perturbation is damped, and the system goes back to its control trajec-

tory $\mathbf{X}(t)$; in this case, the trajectory is *asymptotically stable*. If at least one $w_i(t, \tau)$ stays strictly positive, perturbations exist that will modify durably the trajectory. In our case, no trajectory is asymptotically stable since the model is insensitive to any perturbation that corresponds to a change in monetary unit. As a consequence, this specific perturbation is not damped and it can be associated, without loss of generality, to $w_1(t, \tau) = 1$ at all times $t > \tau$; this type of behavior is called *neutral stability*, *i.e.* perturbations in this phase-space direction neither grow nor decay.

From the w_i , we can now calculate the Lyapunov exponents Λ_i through the relation:

$$\Lambda_i = \lim_{t \rightarrow +\infty} \frac{1}{t} \ln(w_i(t, \tau)) . \quad (29)$$

If at least one Λ_i is strictly positive, there are perturbations that increase exponentially with time and the system exhibits *sensitivity to its initial state*. In this latter case, the model is chaotic.

We calculated the Lyapunov exponents of NEDyM for three values of the investment flexibility: $\alpha_{inv} = 0.1$ for which the model is stable; $\alpha_{inv} = 2$, for which the model tends to a limit cycle; and $\alpha_{inv} = 10$, for which the model exhibits complex dynamics.

For $\alpha_{inv} = 0.1$, only $\Lambda_1 = 0$, corresponding to the $w_1 = 1$ discussed above, and the other seven Λ_i are strictly negative. For $\alpha_{inv} = 2$, there is a second $\Lambda_i = 0$, which corresponds to the tangent direction along the limit cycle.

For $\alpha_{inv} = 10$, the four largest Lyapunov exponents are strictly positive and lie between 0.09 and 0.11 yr^{-1} . Hence infinitesimal perturbations exist that make the trajectory diverge with a characteristic time of about 10 years. The model is therefore chaotic in this case and no economic forecast would be able to provide an accurate and reliable prediction over more than 10 years with this value of the investment flexibility.

5.4 Power spectrum

To study further the changes in model behavior that are apparent in Figs. 2 and 3, we computed the power spectrum of several model variables, for many values of the key parameter α_{inv} . The results are plotted in Fig. 6 for one of the diagnostic variables, production Y , and for five values of α_{inv} ; results for other model variables and for intermediate values of α_{inv} are quite similar (not shown).

When investment flexibility is low, this figure shows an oscillation with a

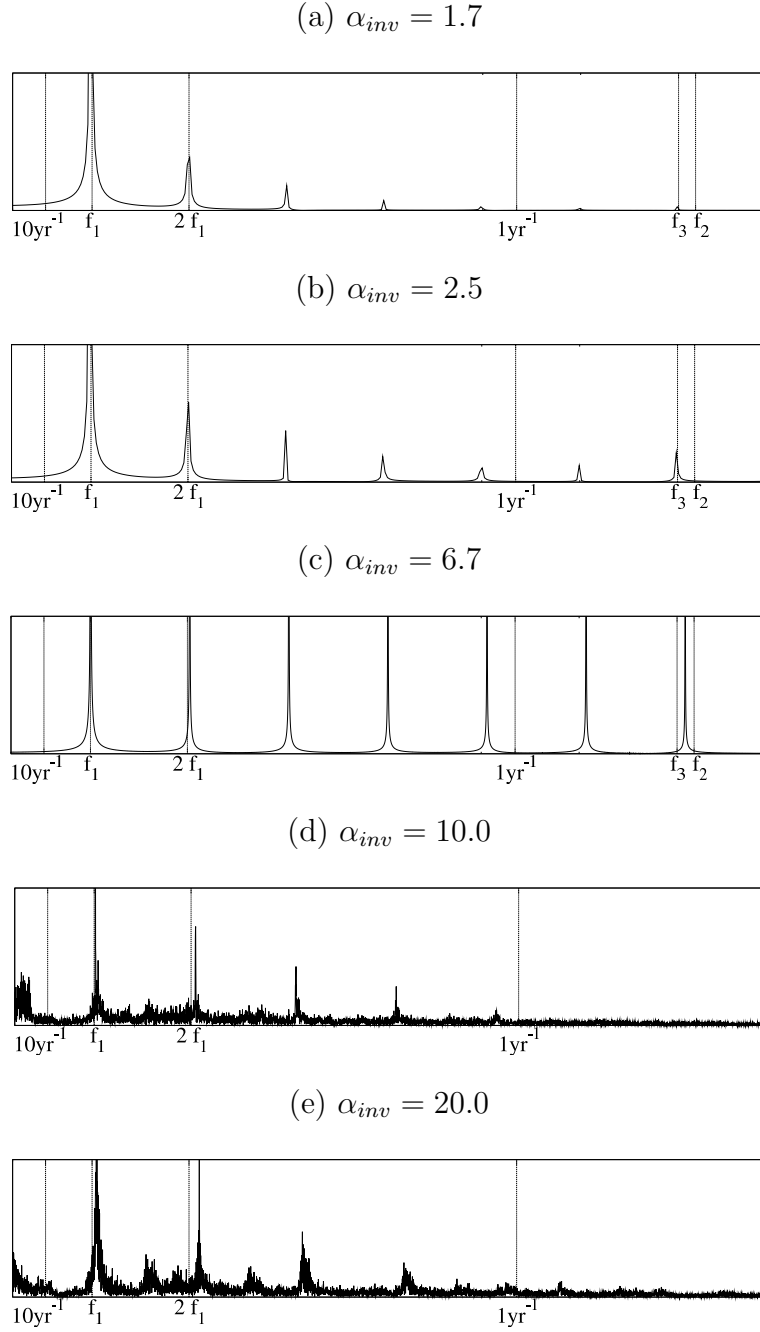


Fig. 6. Power spectrum of production evolution, $Y = Y(t)$, for five values of $\alpha_{inv} = (1.7, 2.5, 6.7, 10.0, 20.0)$, with arbitrary vertical units. Calculated by Fast Fourier Transform (FFT) from the model limit cycle, with time series of at least 3000 years. When necessary, the frequency of the seventh harmonic of the main frequency is indicated as f_3 , and the frequency of the second pair of eigenvalues as f_2 .

period of $T_1 = 5.4$ yr, which is close to the period given by the first pair of eigenvalues, and a few harmonics of decreasing amplitude. No significant variability is found for periods larger than 6 years. The period, as well as the amplitude of nonlinear oscillations depends on the parameters; this explains why the observed main period $T_1 = 5.4$ yr differs slightly from its value at the bifurcation point, $\alpha_{inv} = 1.39$, namely $T' = 5.5$ yr.

As investment flexibility increases, so does the range of oscillations in most variables (see again Figs. 2 and 3). Moreover, the spectrum becomes richer in high-frequency peaks, due to the presence of harmonics of the main frequency, $f_1 = 1/5.4 \text{ yr}^{-1}$.

An interesting phenomenon of frequency locking (Winfree, 1980; Ghil, 1994) occurs in NEDyM: for $\alpha_{inv} = 1.7$ and $\alpha_{inv} = 2.5$, the frequency of the second pair of eigenvalues $f_2 = 1/276 \text{ dy}^{-1}$, does not appear at all in the power spectrum. The seventh harmonic of the main frequency, $f_3 = 7f_1 = 7/5.4 \text{ yr}^{-1} = 1/283 \text{ dy}^{-1}$, however, is very close to it, and exhibits a larger amplitude than previous harmonics, suggesting that the second frequency f_2 is practically locked on the seventh harmonic of the main one, at the frequency f_3 . As already stated in section 4.2, this latter frequency may also be locked in the real world on the annual economic cycle, amplifying the amplitude of the latter (Jin et al., 1994, 1996; Tziperman et al., 1994). For $\alpha_{inv} = 6.7$, the locking is weaker, and the peak lies halfway between f_3 and f_2 .

When investment flexibility is very high, $\alpha_{inv} \geq 10.0$, strong variability on even longer time scales appears. This variability is accompanied by the appearance of irregular, deterministically chaotic behavior; it is also visible in Fig. 3 and is associated with the constraints $\gamma_{min} \leq \Gamma_{inv} \leq \gamma_{max}$ becoming effective. This highly nonlinear, saturation effect arises, as described in Section 5.1, from the much larger variability, and gives rise, in turn, to the continuous portion of the spectrum apparent in Figs. 6d and 6e.

6 Concluding remarks

6.1 Summary

In this paper, we pursued an endogenous approach to modeling economic cycles, defined as “*serially correlated movements about trend which are not caused by movements in the availability of factors of production*” (Lucas, 1975). In NEDyM, business cycles arise from inertial effects (see Sections 2, 3 and 4.2), namely from delays in the *adjustment* of price, wages and investment decisions, even though the model’s long-term equilibrium is neo-classical

in nature.

The business cycles generated by this model, subject to an assumption of managerial control of investment decisions, reproduce several key features of observed cycles: (i) their four phases, namely (a) recovery, (b) expansion with inflation, full employment and decrease in the profit rate, (c) contraction with deflation, and (d) depression, unemployment and restoration of the profit rate; (ii) the typical leads and lags among major economic indicators (Zarnovitz, 1985; King and Watson, 1996); (iii) the mean period of 5–6 years; and (iv) the cycle’s asymmetry (Kontolemis, 1997), with a longer expansion phase and a more rapid contraction.

The key trigger of these cycles is basically the multiplier-accelerator effect that occurs when investment flexibility exceeds a certain threshold. The instability that leads to the cycles is indeed generated by the interplay between short-term Keynesian effects, with delays in price adjustments, on the one hand, and supply-side and labor-availability constraints, on the other: as investment increases, the demand increases more rapidly than the production capacity and the employment rate reaches high values, leading to inflation of price and wages, reduced profits per capital unit, and thus reduced investment. The interaction of these processes creates an oscillatory behavior, whose amplitude increases with the investment flexibility.

In our model, however, fluctuations in demand do not arise from a Keynesian hoarding of money but from the nonproductive, closed money loop between consumer savings and dividends that sets in when a low profit rate discourages producers from investing. The cycles’ asymmetry in NEDyM is due to the larger elasticity of prices with respect to the level of goods inventory during the contraction phase, when demand is low, than during the expansion phase, when demand is large.

When investment flexibility increases, the amplitude of the business cycle is greater and financial constraints on investment come into play; investment cannot keep responding to the profit signal and this newly binding constraint generates complex dynamics and chaotic behavior. This type of behavior is most likely to be observed at times when several resource constraints become also binding; along with financial constraints, these can include limitation on natural resources or labor availability.

Our model results do not pretend to resolve the controversy between those who affirm (Baumol and Benhabib, 1989) and those who deny (Brock et al., 1991) the existence of chaos in macroeconomic data. We agree, in fact, with those who think that chaotic behavior in economic series might not be detectable by time series analysis alone, because of the shortness of the series (Jarsulic, 1993); compare similar statements for climatic time series in Grass-

berger (1986), Ghil (1994) and Ghil et al. (2002). We simply meant to provide a consistent framework for discussing the various views of long-term economic dynamics and how they may change the assessment of policies proposed to cope with the issues of climate change or energy transitions⁹.

Our results suggest the existence of a trade-off between adaptability and stability: if investment flexibility is low, there are no business cycles, or only small-amplitude ones, but the economic adaptation to exogenous shocks may be slow and inefficient. If investment flexibility is high, the economy may adapt more quickly to exogenous shocks but the population's welfare is affected by large endogenous fluctuations in employment and by mean-production losses due to nonlinear rectification mechanisms. To what extent these results may change when considering a shareholder-controlled economy has to be investigated systematically in terms of such an economy's cyclical behavior and return to steady state after an external shock. These questions will be pursued in a follow-up paper.

6.2 Discussion

Introducing various types of inertia into the Solow model allowed us to gain useful insights into economic dynamics. Indeed, while Solow (1956) showed that perfect labor–capital substitution permits the economy to follow stable growth pathways, we suggest that, even with factor substitution, inertia and adjustment delays may destabilize these pathways, leading to short-term fluctuations or even chaotic behavior. The consequence is that, for a simple economy without endogenous technical progress and demographic growth, complex behavior in the economic system can be described through deterministic endogenous factors, instead of the random exogenous shocks of “*real-cycle*” theory (Kydland and Prescott, 1982). Moreover, nonlinear rectification mechanisms make it impossible to model the economy through relationships between time-averaged values, as can be done in real-cycle theory: changes in the short-term dynamics do modify the averaged value of the variables along the trajectory, even though the latter evolves around the unchanged, but unstable, equilibrium.

The blueprint for further economic research suggested by our results seems straightforward. For short- and medium-term dynamics, we may have to incorporate some of the Keynesian effects driven by actual quantities (*e.g.*, Chiarella and Flaschel, 2000; Asada et al., 2004). For the long-term, we have first to explore the consequences of various shifts in corporate behavior, between managerial and shareholder types of control. Second, we have to consider the in-

⁹ See for instance the preliminary results on the economic effects of extreme events in Hallegatte (2005).

teractions between business cycles and induced technical change within an endogenous-growth framework (*e.g.*, Schumpeter, 1939; Aghion and Howitt, 1998), and examine the possibility of multiple long-term equilibria.

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